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LAUW.HTML   1 November 1999   David Halprin davrin@net2000.com.au

FRACTALS AND THEIR BASINS OF STABILITY

Hans Lauwerier outlined a method to predict the Cardioid basin arising from the iteration of the Circle in the Mandelbrot fractal, however he left it at that one example, therefore the generalisation of the approach begs investigation.

1) $z_1 = F(z) = z^2 + c$

2) $F(z) \approx F(z_0) + (z - z_0).F'(z_0)$

where $F(z_0)$ represents a translation, $(z - z_0)$ represents a rotation and $F'(z_0)$ represents a scale factor for enlargement, provided it is non-zero.

The iterative process of fractal generation depends on

3) $z_{n+1} = F(z_n)$

We have to consider the fixed (equilibrium) points.

If, at a fixed point z, $|F'(z)| < 1$ then the orbit of z_0 in the vicinity of z approaches a stable (attracting) equilibrium point.

If however $|F'(z)| > 1$ then the orbit is unstable and is repelled by z_0.

The other case to be considered is when $|F'(z)| = 1$ where we have a neutral equilibrium.

We put $z_1 = z$ in 1) and solve

4) $z = z^2 + c$

5) $\alpha_1, \alpha_2 = -½ \mp \sqrt{¼ - c}$

6) $\alpha_1 + \alpha_2 = 1$

Graphically, the mid point of the two roots is ½. The stability of the orbit depends on

7) $|F'(z)| = 2z$ Since $\alpha_2 > ½$ it creates an unstable orbit so we discard it from further consideration, while $\alpha_1 < ½$ creates a stable orbit.

At the margin of the area of stability, (the basin),

8) $|2\alpha_1| = 1$

9) $2\alpha_1 = \cos\theta + i.\sin\theta = e^{i\theta} = 1 - \sqrt{1 - 4c}$

therefore

10) $1 - 4c = 1 - 2e^{i\theta} + e^{2i\theta}$

11) $c = ¼\left(e^{2i\theta} - 2e^{i\theta}\right)$ **Cardioid**

12) $a = -¼(\cos 2\theta - 2\cos\theta)$; $b = -¼(\sin 2\theta - 2\sin\theta)$

Mandelbrot, no doubt, was aware that the complex equation for a circle can be generalised for `n', any positive integer exponent of z, yet all circles are concentric and coincident, with radius = 1:

$f(z_1) = z^n + c$ His choice of n = 2 was the lowest value for n to produce a fractal, since n = 1 produces only one circular basin.

Jumping ahead, one finds that all the basins for the areas of stability for the total of (n-1) fractals can be approximated to a high degree of accuracy by $c = k.\left(e^{ni\theta} - ne^{i\theta}\right)$, where `k' is an arbitrary constant, (They were generated by Derive XM, see .GIF files below).

Further, each of the basins has (n-1) cusps starting with one cusp in the normal cardioid where n = 2. The background colour is contained within an (n-1)-sided shape also, when iterated by FRACTINT, (see .GIF files below), a freeware software program, readily available for download from the web.

However the precise value for `c' can be calculated for n = 3, while the higher values of n seem to be insoluble in general terms.

13) $z_1 = F(z) = z^3 + c$

14) $F'(z) = 3z^2$

15) $z^3 - z + c = 0$

16) $\alpha_1 = \dfrac{2}{\sqrt{3}}.\sin\left(⅓.\arcsin\dfrac{3\sqrt{3}c}{2}\right)$

17) $\alpha_2 = -\dfrac{2}{\sqrt{3}}.\sin\left(⅓.\arcsin\dfrac{3\sqrt{3}c}{2}\right) + \dfrac{\pi}{3}$

18) $\alpha_3 = \dfrac{2}{\sqrt{3}}.\cos\left(⅓.\arcsin\dfrac{3\sqrt{3}c}{2}\right) + \dfrac{\pi}{6}$

19) $|F'(z)| = 3\alpha_1^2 = e^{i\theta} \quad \therefore \quad e^{\frac{i\theta}{2}} = 2.\sin\left(⅓.\arcsin\dfrac{3\sqrt{3}c}{2}\right)$

20) $\arcsin\left(\dfrac{e^{\frac{i\theta}{2}}}{2}\right) = ⅓.\arcsin\dfrac{3\sqrt{3}c}{2}$

21) $c = \dfrac{2}{3\sqrt{3}}.\sin\left(3.\arcsin\dfrac{e^{\frac{i\theta}{2}}}{2}\right)$

22) $c = k.\left(\dfrac{e^{\frac{i\theta}{2}}}{2}.\left(3 - e^{i\theta}\right)\right)$

We take the first of the three solutions, since the other two do not produce stable orbits.

Equations 21) and 22) are alternative expressions for plotting the double cusped \`cardioid'.

23) $F(z) = z^4 + c \quad , \quad F'(z) = 4z^3$

24) $z^4 - z + c = 0 \quad , \quad |F'(z)| = |4\alpha_1^3| = e^{i\theta}$

25) $c = k.\left(e^{4i\theta} - 4e^{i\theta}\right)$

26) $F(z) = z^n + c \quad , \quad F'(z) = n.z^{n-1}$

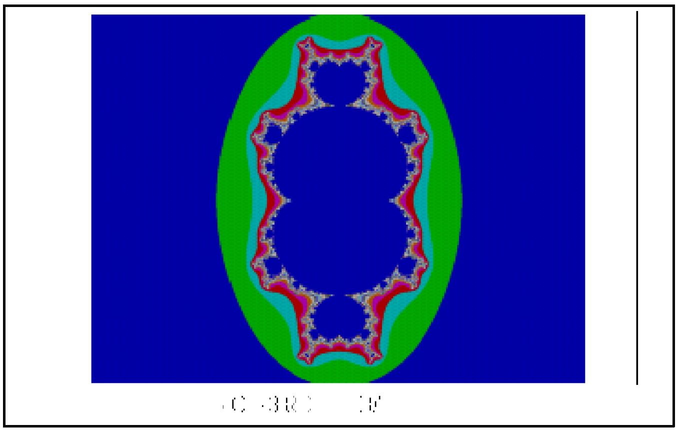
27) $z^n - z + c = 0 \quad , \quad |F'(z)| = |n.\alpha^{n-1}| = e^{i\theta}$

28) $c = k.\left(e^{ni\theta} - ne^{i\theta}\right)$

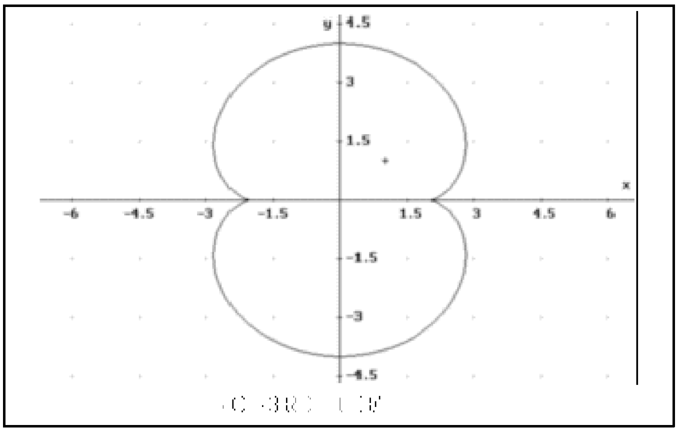
Equation 25) is taken from the general form, since the closed form solutions for \`z' in terms of \`c' cannot be solved exactly for \`c' in terms of \`z'.

The reciprocal question then arises:- "Can we nominate a basin of stability for a conjectured fractal and do the maths to find which

curve(s), when iterated, will produce it?"



CIRC-3RD.GIF



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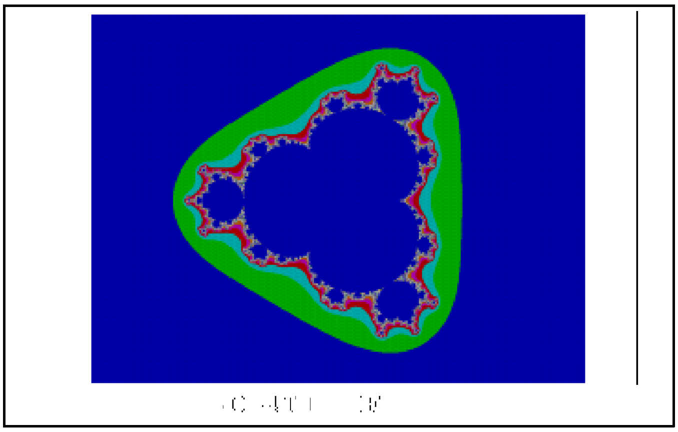

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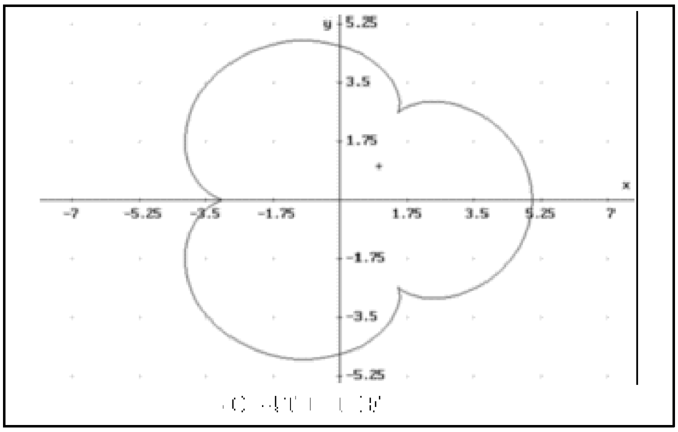

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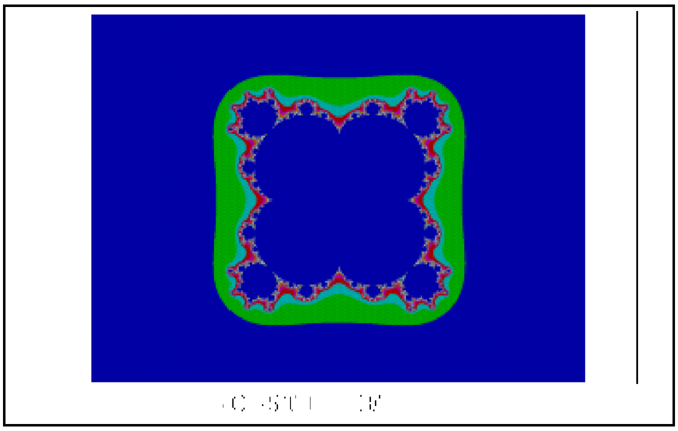

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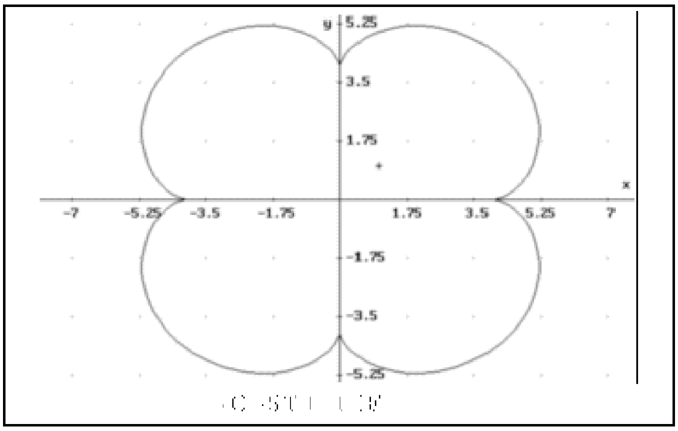

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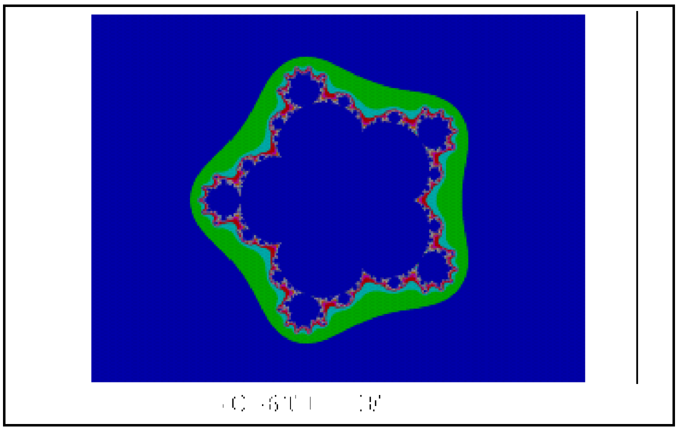
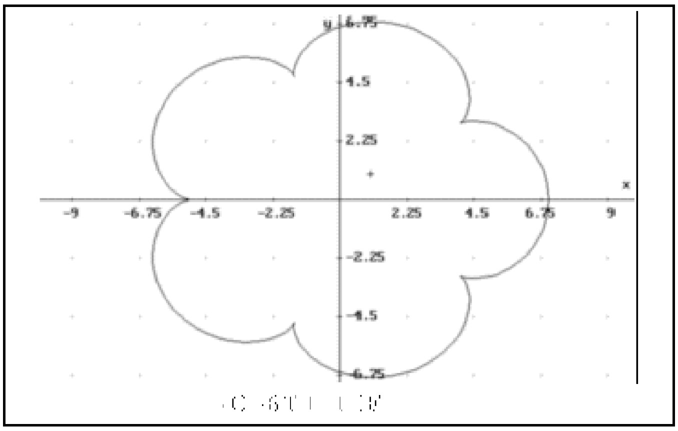
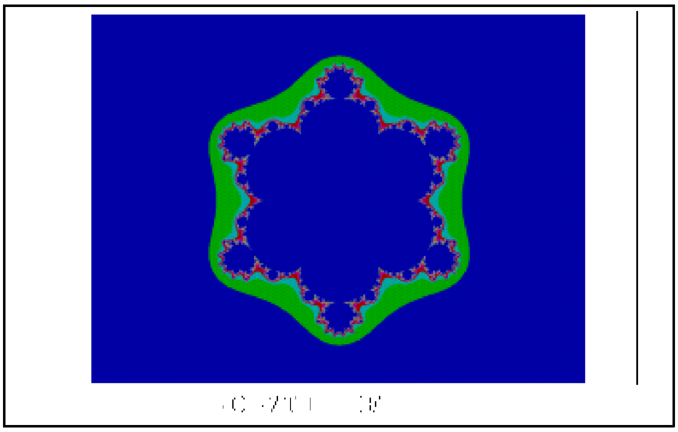
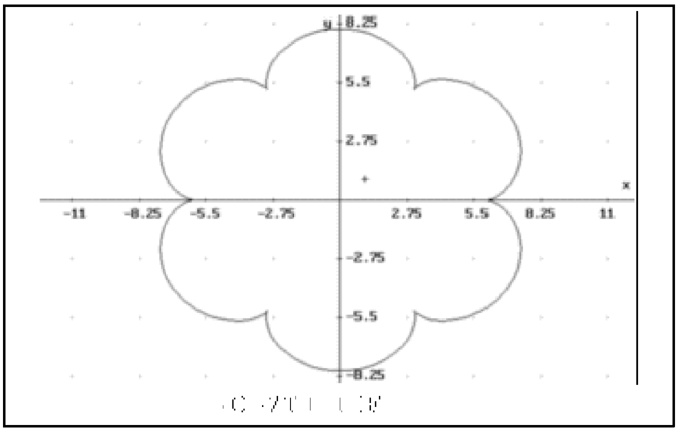




CIRC-6TH.GIF

CIRC-6TH{0}.GIF





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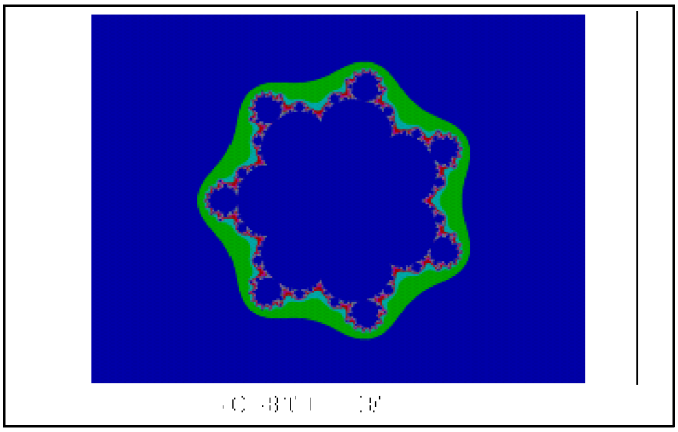

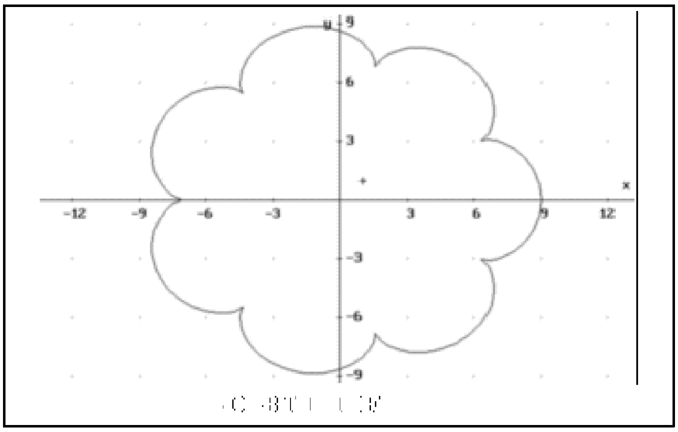

CIRC-8TH.GIF

CIRC-8TH{0}.GIF

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